Nonresonance optical breathers in nonlinear and dispersive media

G. T. Adamashvili
Tbilisi State University, Tbilisi, Georgia 380079

A. A. Maradudin
Department of Physics and Astronomy, University of California, Irvine, California 92697

(Received 8 July 1996)

A theory of nonresonance optical breathers in nonlinear and dispersive media is developed. The optical wave equation with a damping term can be solved by using the reductive-perturbation method. Explicit analytic expressions for the parameters of these nonlinear waves are obtained. The stability of a breather in the presence of its interaction with impurity-resonance atoms and a finite conductivity is also discussed in detail. [S1063-651X(97)08106-3]

PACS number(s): 42.65.Tg

I. INTRODUCTION

The propagation of an optical wave in a medium is accompanied by various changes of its form. The main mechanisms that change the forms of waves are dispersion and, if the amplitude of a wave is finite, nonlinearity. The most interesting are those wave processes for which the effects distorting the form of the wave compensate each other exactly. Under these conditions nonlinear waves of an invariant profile are formed. Among nonlinear waves of a stationary form, solitons and breathers are very often encountered. The propagation of these waves displays its own specific features. In nonlinear physical theories they play as fundamental a role as harmonic oscillations do in the linear wave theory. The conditions for the existence of solitons and breathers are different. A breather is the bound state of a soliton and an antisoliton. It possesses an internal structure. Such a formation is unstable if its energy exactly equals the sum of the energies of a separated soliton and antisoliton, i.e., if the binding energy is zero. This condition is not stable, in that even a small perturbation leads to decay of the bound state into a separate soliton and antisoliton whose velocities are proportional to the perturbation and generally differ. Exactly such a situation is realized, for example, for the breather of the nonlinear Schrödinger equation (NSE), where at the same time a soliton solution of this equation is quite stable. Consequently, in one and the same system the existence of solitons does not assure the existence of breathers, and vice versa. Unlike solitons, breathers can be excited for relatively small amplitudes (energies) or areas of pulses [1–5]. They arouse particular interest, because breather solutions of some equations of nonlinear optics are highly stable. The determination of the mechanisms causing the formation of breathers and the study of influences leading to a variation of their parameters are among the principal problems of nonlinear waves. The mechanisms of formation of breathers are varied and depend on the properties of the medium in which an optical wave propagates, as well as on the parameters of the wave.

In recent years there has been growing interest in the analysis and characterization of nonlinear dispersive systems in optics [6–8]. In the propagation of a pulse in a dispersive medium, its shape will not remain unchanged: its width will spread. This is due to the fact that waves of different wavelengths propagate in a dispersive medium with different velocities.

On the other hand, the effects of nonlinearity lead to a progressive deformation of the initial pulse, which increases with increasing time. As a result of the competition between the nonlinearity, which increases the curvature of the profile of the pulse, and the dispersion, which causes the profile to broaden, the shape of the nonlinear wave is stabilized, a breather state is formed.

The basic sources of the optical nonresonance nonlinearity in solids may be the following. The medium possesses a nonlinear susceptibility, principal of which are nonlinearities of second (quadratic) and third (cubic) orders. In the case of quadratic nonlinearity, the interaction of wave packets of different frequencies can be observed, but in the case of cubic nonlinearity (Kerr-type nonlinearity) self-influence of waves take place.

Different features of solitons in these systems have been investigated in detail. However, a whole class of interesting nonlinear phenomena, such as the mechanisms of formation of optical breathers and the effects leading to their instability or changes of their parameters (for example, interaction with free electrons, resonance-impurity atoms, and others), have not, to our knowledge, been investigated in these systems.

The processes of formation of nonresonance breathers in nonlinear and dispersive media and the stability of a breather in the presence of its interaction with impurity-resonance atoms and a finite conductivity are also discussed in this paper.

II. BASIC EQUATION

Consider the mechanism of the formation of a nonlinear optical wave in a nonlinear and dispersive medium in the case where an optical pulse of width $T \ll T_{1,2}$, with wave vector k and frequency $\omega \gg T^{-1}$, propagates in the positive direction along the z axis, where T_1 and T_2 are the longitudinal and transverse relaxation times of impurity atoms. Without specifying the physical nature of the dispersive process, we describe the dependence of the dielectric tensor ϵ_{ij} on two variables—the wave vector k and frequency ω of the wave [7,8]. We will investigate the case where the medium is isotropic, $\epsilon_{ij}=\epsilon_0\delta_{ij}$, where δ_{ij} is the Kronecker symbol and ϵ_0 is the dielectric constant. The wave equation for the x component of the strength of the electric field E in this case reduces to the form

$$c^2\frac{\partial^2 E}{\partial z^2}=\frac{\partial^2 D}{\partial t^2}, \tag{1}$$

where c is the speed of light in vacuum,

$$D=D_l+4\pi P \tag{2}$$

is the x component of the electric displacement vector,

$$D_l(z,t)=\int \epsilon_0(z_1,t_1)E(z-z_1,t-t_1)dz_1dt_1 \tag{3}$$

is the linear part of the quantity D, and

$$P=P^{(2)}+P^{(3)} \tag{4}$$

is the nonlinear part of the polarization. $P^{(2)}$ and $P^{(3)}$ are nonresonant nonlinear polarizations of the second and third orders:

$$P^{(2)}=\int \chi(z,t,x_1,x_2,t_1,t_2)E(z-x_1,t-t_1)\times E(z-x_1-x_2,t-t_1-t_2)dt_1dt_2dx_1dx_2, \tag{5}$$

$$P^{(3)}=\int \rho(z,t,x_1,x_2,x_3,t_1,t_2,t_3)E(z-x_1,t-t_1)\times E(z-x_1-x_2,t-t_1-t_2)\times E(z-x_1-x_2-x_3,t-t_1-t_2-t_3)\times dx_1dx_2dx_3dt_1dt_2dt_3, \tag{6}$$

where χ and ρ are the quadratic and cubic susceptibility tensors. The nonlinear optical response characterized by the parameters χ and ρ leads to many interesting phenomena. Nonlinearity of the second order is responsible for second harmonic generation (doubling of the frequency), for the generation of sum and difference frequencies, for parametric amplification, and for other effects caused by processes of the three-frequency interaction of the waves. Nonlinearity of the third order reduces to effects of the four-frequency interaction of the waves, such as self-focusing, Raman scattering, and others. Note that we do not consider here the physical origin of the nonlinear coefficients χ and ρ; we regard them as material parameters and study the nonlinear optical phenomena to which they give rise [7,8].

Substituting Eqs. (3)–(6) into Eq. (2), we obtain the following expression:

$$D(z,t)=\int \epsilon_0(z_1,t_1)E(z-z_1,t-t_1)dz_1dt_1+4\pi\int \chi(z,t,x_1,x_2,t_1,t_2)E(z-x_1,t-t_1)E(z-x_1-x_2,t-t_1-t_2)\times dt_1dt_2dx_1dx_2+4\pi\int \rho(z,t,x_1,x_2,x_3,t_1,t_2,t_3)E(z-x_1,t-t_1)E(z-x_1-x_2,t-t_1-t_2)\times E(z-x_1-x_2-x_3,t-t_1-t_2-t_3)dx_1dx_2dx_3dt_1dt_2dt_3. \tag{7}$$

By combining Eqs. (7) and (1), we obtain the following nonlinear wave equation for E:

$$-c^2\frac{\partial^2 E}{\partial z^2}+\frac{\partial^2}{\partial t^2}\left\{\int \epsilon_0(z_1,t_1)E(z-z_1,t-t_1)dz_1dt_1+4\pi\int \chi(z,t,x_1,x_2,t_1,t_2)E(z-x_1,t-t_1)E(z-x_1-x_2,t-t_1-t_2)\times dt_1dt_2dx_1dx_2+4\pi\int \rho(z,t,x_1,x_2,x_3,t_1,t_2,t_3)E(z-x_1,t-t_1)E(z-x_1-x_2,t-t_1-t_2)\times E(z-x_1-x_2-x_3,t-t_1-t_2-t_3)dx_1dx_2dx_3dt_1dt_2dt_3\right\}=0. \tag{8}$$

We can simplify Eq. (8) significantly. For this purpose, we represent the function E in the form

$$E=\sum_l \hat{E}_l Z_l, \tag{9}$$

where $\hat{E}_l$ are slowly varying complex amplitudes of the optical wave $Z_l=e^{il(kz-\omega t)}$ and l runs through the values ± 1, $\pm 2,\ldots$. To guarantee the reality of the quantity E, we set $\hat{E}_l=\hat{E}^*_{-l}$. We note that such a representation of the solution of a nonlinear wave equation has been widely used in the theory of nonlinear waves [7–13]. This approximation is based on the assumption that the envelopes $\hat{E}_l$ vary slowly in space and time as compared with the carrier wave parts, i.e.,

$$\left|\frac{\partial \hat{E}_l}{\partial t}\right| \ll \omega |\hat{E}_l|, \quad \left|\frac{\partial \hat{E}_l}{\partial z}\right| \ll k |\hat{E}_l|,$$

and is called the slowly varying envelope approximation.

On substituting Eq. (9) into Eq. (8), we obtain

$$\sum_{l=\pm 1} Z_l \Bigg\{ \left(W_l + i\beta_0 \frac{\partial}{\partial t} - i\alpha_0 \frac{\partial}{\partial z} - \mu_0 \frac{\partial^2}{\partial z^2} - \delta_0 \frac{\partial^2}{\partial t^2} - \gamma_0 \frac{\partial^2}{\partial t \partial z} \right) \hat{E}_l + 4\pi \sum_{l',l''=\pm 1} l^2 \omega^2 \Bigg[-\chi_{l,l'} \hat{E}_{l-l'} \hat{E}_{l'} + iA_1 \frac{\partial \hat{E}_{l-l'}}{\partial z} \hat{E}_{l'}$$
$$+ i(A_1 + A_2) \hat{E}_{l-l'} \frac{\partial \hat{E}_{l'}}{\partial z} - i\left(B_1 + 2\frac{\chi_{l,l'}}{l\omega} \right) \frac{\partial \hat{E}_{l-l'}}{\partial t} \hat{E}_{l'} - i\left(B_1 + B_2 + 2\frac{\chi_{l,l'}}{l\omega} \right) \hat{E}_{l-l'} \frac{\partial \hat{E}_{l'}}{\partial t} - \rho \hat{E}_{l-l'-l''} \hat{E}_{l'} \hat{E}_{l''}$$
$$+ ia_1 \frac{\partial \hat{E}_{l-l'-l''}}{\partial z} \hat{E}_{l'} \hat{E}_{l''} + i(a_1 + a_2) \hat{E}_{l-l'-l''} \frac{\partial \hat{E}_{l'}}{\partial z} \hat{E}_{l''} + i(a_1 + a_2 + a_3) \hat{E}_{l-l'-l''} \hat{E}_{l'} \frac{\partial \hat{E}_{l''}}{\partial z} - i\left(b_1 + 2\frac{\rho}{l\omega} \right) \frac{\partial \hat{E}_{l-l'-l''}}{\partial t} \hat{E}_{l'} \hat{E}_{l''}$$
$$- i\left(b_1 + b_2 + 2\frac{\rho}{l\omega} \right) \hat{E}_{l-l'-l''} \frac{\partial \hat{E}_{l'}}{\partial t} \hat{E}_{l''} - i\left(b_1 + b_2 + b_3 + 2\frac{\rho}{l\omega} \right) \hat{E}_{l-l'-l''} \hat{E}_{l'} \frac{\partial \hat{E}_{l''}}{\partial t} \Bigg] \Bigg\} = 0, \tag{10}$$

where

$$W_l = l^2(c^2k^2 - \omega^2\kappa_l), \quad \alpha_0 = l(2kc^2 - l\omega^2 A_l), \quad \beta_0 = -l\omega(2\kappa_l + l\omega B_l'),$$

$$\gamma_0 = l\omega(2A_l' + l\omega T_l), \quad \delta_0 = -(l^2\omega^2 D_l + \kappa_l + 2l\omega B_l'), \quad \mu_0 = c^2 - \omega^2 l^2 C_l,$$

$$\kappa_l = \epsilon_0(lk, l\omega) = \int \epsilon_0(z,t) e^{il(\omega t - kz)} dt dz, \quad A_l' = \frac{\partial \kappa_l}{\partial(lk)},$$

$$B_l' = \frac{\partial \kappa_l}{\partial(l\omega)}, \quad C_l = \frac{1}{2}\frac{\partial^2 \kappa_l}{\partial(lk)^2}, \quad D_l = \frac{1}{2}\frac{\partial^2 \kappa_l}{\partial(l\omega)^2}, \quad T_l = \frac{\partial^2 \kappa_l}{\partial(lk)\partial(l\omega)},$$

$$\chi_{l,l'} = \int \chi(z-x, z-y, t-t_1, t-t_2) e^{-il(kx - \omega t_1) - il'(ky - \omega t_2)} dx dy dt_1 dt_2,$$

$$A_1 = \frac{\partial \chi_{l,l'}}{\partial(lk)}, \quad A_2 = \frac{\partial \chi_{l,l'}}{\partial(l'k)}, \quad B_1 = \frac{\partial \chi_{l,l'}}{\partial(l\omega)}, \quad B_2 = \frac{\partial \chi_{l,l'}}{\partial(l'\omega)},$$

$$\rho = \rho_{l,l',l''} = \int \rho(z,t,x_1,x_2,x_3,t_1,t_2,t_3) e^{-il(kx_1 - \omega t_1) - i(l'+l'')(kx_2 - \omega t_2) - il''(kx_3 - \omega t_3)} dx_1 dx_2 dx_3 dt_1 dt_2 dt_3,$$

$$a_1 = \frac{\partial \rho}{\partial(lk)}, \quad b_1 = \frac{\partial \rho}{\partial(l\omega)},$$

$$a_2 = \frac{\partial \rho}{\partial[(l'+l'')k]}, \quad b_2 = \frac{\partial \rho}{\partial[(l'+l'')\omega]},$$

$$a_3 = \frac{\partial \rho}{\partial(l''k)}, \quad b_3 = \frac{\partial \rho}{\partial(l''\omega)}.$$

The analysis of Eq. (10) can be carried out by two different methods, depending on whether we investigate the problem of the evolution of the initial perturbation (Case I) or we consider the propagation in the medium of a pulse, which is specified on the boundary of the medium (Case II). Although the corresponding equations appear different, we must note that in some sense they are identical to each other.

In the first case, the quantities $\hat{E}_l$ can be represented as [1–4,14,15]

$$\hat{E}_l(z,t)=\sum_{\alpha=1}^{+\infty}\sum_{n=-\infty}^{+\infty}\epsilon^{\alpha}Y_n\hat{\varphi}_{l,n}^{(\alpha)}(\zeta,\tau), \tag{11}$$

where

$$Y_n=e^{in(Qz-\Omega t)},\quad \zeta=\epsilon Q(z-v_g t),\quad \tau=\epsilon^2 t,\quad v_g=\frac{d\Omega}{dQ},$$

and ϵ is the small parameter.

In the second case, we can represent the quantity $\hat{E}_l$ as

$$\hat{E}_l(z,t)=\sum_{\alpha=1}^{+\infty}\sum_{n=-\infty}^{+\infty}\epsilon^{\alpha}X_n f_{l,n}^{(\alpha)}(\xi,\nu), \tag{12}$$

where

$$X_n=\epsilon^{in(\widetilde{Q}z-\widetilde{\Omega}t)},\quad \xi=\epsilon\left(t-\frac{z}{u}\right),\quad \nu=\epsilon^2 z,\quad u=\left(\frac{d\widetilde{Q}}{d\widetilde{\Omega}}\right)^{-1}.$$

Such a representation allows us to separate from $\hat{E}_l$ the still more slowly changing quantities $\varphi_{l,n}^{(\alpha)}$ and $f_{l,n}^{(\alpha)}$. Consequently, it is assumed that the quantities Ω, Q, $\varphi_{l,n}^{(\alpha)}$, and $f_{l,n}^{(\alpha)}$ satisfy the inequalities $\omega\gg\Omega$, $k\gg Q$, $\omega\gg\widetilde{\Omega}$, $k\gg\widetilde{Q}$, $|\partial\varphi_{l,n}^{(\alpha)}/\partial t|\ll\Omega|\varphi_{l,n}^{(\alpha)}|$, $|\partial\varphi_{l,n}^{(\alpha)}/\partial z|\ll Q|\varphi_{l,n}^{(\alpha)}|$, $|\partial f_{l,n}^{(\alpha)}/\partial t|\ll\widetilde{\Omega}|f_{l,n}^{(\alpha)}|$, $|\partial f_{l,n}^{(\alpha)}/\partial z|\ll\widetilde{Q}|f_{l,n}^{(\alpha)}|$.

III. BREATHER SOLUTION OF EQ. (10) IN THE FIRST CASE

We begin by considering the solution of Eq. (10) in Case I, i.e., an initial-value problem. In this analysis we use the expansion (11). On substituting it into Eq. (10), we obtain the equation

$$\begin{aligned}\sum_{\alpha,l,n}\epsilon^{\alpha}Z_lY_n\Bigg\{&\left[\widetilde{W}_{l,n}+\epsilon J_{l,n}\frac{\partial}{\partial\zeta}+\epsilon^2H_{l,n}\frac{\partial^2}{\partial\zeta^2}+\epsilon^2h_{l,n}\frac{\partial}{\partial\tau}+O(\epsilon^3)\right]\varphi_{l,n}^{(\alpha)}\\&-\sum_{\alpha',l',n'}\epsilon^{\alpha'}\left[F_{l,n,l',n'}\varphi_{l-l',n-n'}^{(\alpha)}\varphi_{l',n'}^{(\alpha')}-i\epsilon\left(f_{l,n,l',n'}\frac{\partial\varphi_{l-l',n-n'}^{(\alpha)}}{\partial\zeta}\varphi_{l',n'}^{(\alpha')}+\widetilde{f}_{l,n,l',n'}\varphi_{l-l',n-n'}^{(\alpha)}\frac{\partial\varphi_{l',n'}^{(\alpha')}}{\partial\zeta}\right)\right]\\&-\sum_{\alpha',l',n',\alpha'',l'',n''}\epsilon^{\alpha'+\alpha''}\lambda_{l,n,l',n',l'',n''}\varphi_{l-l'-l'',n-n'-n''}^{(\alpha)}\varphi_{l'n'}^{(\alpha')}\varphi_{l'',n''}^{(\alpha'')}\Bigg\}=0,\end{aligned} \tag{13}$$

where

$$\begin{gathered}\widetilde{W}_{l,n}=W_l+\alpha Q+\beta\Omega-\gamma\Omega Q+\delta\Omega^2+\mu Q^2,\\ J_{l,n}=\frac{Q}{in}[\alpha+\beta v_g+2Q\mu+2\delta\Omega v_g-\gamma(\Omega+Qv_g)],\\ H_{l,n}=-\frac{Q^2}{n^2}(\mu+\delta v_g^2-\gamma v_g),\quad h_{l,n}=\frac{i}{n}(\beta+2\delta\Omega-\gamma Q),\\ F_{l,n,l',n'}=4\pi l^2\omega^2\left[\chi_{l,l'}+(n-n')QA_1-n'Q(A_1+A_2)+(n-n')\Omega\left(B_1+2\frac{\chi_{l,l'}}{l\omega}\right)+n'\Omega\left(B_1+B_2+2\frac{\chi_{l,l'}}{l\omega}\right)\right],\\ \lambda_{l,n,l',n',l'',n''}=4\pi l^2\omega^2\left\{\rho+n\left(a_1Q+b_1\Omega+\frac{2\Omega}{l\omega}\rho\right)+(a_2Q+b_2\Omega)n'+[(a_2+a_3)Q+(b_2+b_3)\Omega]n''\right\},\\ \alpha=n\alpha_0,\quad\beta=n\beta_0,\quad\gamma=n^2\gamma_0,\quad\delta=n^2\delta_0,\quad\mu=n^2\mu_0.\end{gathered} \tag{14}$$

The explicit forms of the quantities $f_{l,n,l',n'}$ and $f_{l,n,l',n'}$ are not needed because they do not enter into the final results.

To determine the values of $\varphi_{l,n}^{(\alpha)}$, we equate to zero the terms corresponding to like powers of ϵ. As a result, we obtain a chain of equations: in first order in ϵ,

$$\tilde{W}_{l,n}\varphi_{l,n}^{(1)}=0; \tag{15}$$

in second order in ϵ,

$$\tilde{W}_{l,n}\varphi_{l,n}^{(2)}+J_{l,n}\frac{\partial}{\partial\zeta}\varphi_{l,n}^{(1)}-\sum_{l',n'}F_{l,n,l',n'}\varphi_{l-l',n-n'}^{(1)}\varphi_{l',n'}^{(1)}=0; \tag{16}$$

and in third order in ϵ,

$$\tilde{W}_{l,n}\varphi_{l,n}^{(3)}+J_{l,n}\frac{\partial}{\partial\zeta}\varphi_{l,n}^{(2)}+H_{l,n}\frac{\partial^2}{\partial\zeta^2}\varphi_{l,n}^{(1)}+h_{l,n}\frac{\partial}{\partial\tau}\varphi_{l,n}^{(1)}-\sum_{l',n',l'',n''=-\infty}^{+\infty}\Bigg[F_{l,n,l',n'}(\varphi_{l-l',n-n'}^{(2)}\varphi_{l',n'}^{(1)}+\varphi_{l-l',n-n'}^{(1)}\varphi_{l',n'}^{(2)})$$
$$-i\left(f_{l,n,l',n'}\frac{\partial\varphi_{l-l',n-n'}^{(1)}}{\partial\zeta}\varphi_{l',n'}^{(1)}+\tilde{f}_{l,n,l',n'}\varphi_{l-l',n-n'}^{(1)}\frac{\partial\varphi_{l',n'}^{(1)}}{\partial\zeta}\right)-\lambda_{l,n,l',n',l'',n''}\varphi_{l-l'-l'',n-n'-n''}^{(1)}\varphi_{l',n'}^{(1)}\varphi_{l'',n''}^{(1)}\Bigg]=0. \tag{17}$$

In dispersive media, $W_0=W_{\pm1}=0$ and $W_{|l|>1}\neq 0$. In what follows, we shall also be interested in a breather that vanishes at $t\to\pm\infty$. Consequently, according to Eq. (15), only the following terms of all the quantities $\varphi_{l,n}^{(1)}$ differ from zero: $\varphi_{\pm1,\pm1}^{(1)}$ or $\varphi_{\pm1,\mp1}^{(1)}$. Here we consider the situation where $\varphi_{\pm1,\pm1}^{(1)}\neq0$ and $\varphi_{\pm1,\mp1}^{(1)}=0$ in detail. The relation between the quantities Ω and Q, for fixed values of $l=n=\pm1$, is determined from the equation

$$\alpha Q+\beta\Omega-\gamma\Omega Q+\delta\Omega^2+\mu Q^2=0. \tag{18}$$

Substituting Eq. (18) into Eq. (14), we easily see that the following relation holds:

$$J_{\pm1,\pm1}=0. \tag{19}$$

From Eq. (16), we obtain the connection between $\varphi_{\pm2,\pm2}^{(2)}$ and $\varphi_{\pm1,\pm1}^{(1)}$:

$$\varphi_{\pm2,\pm2}^{(2)}=\frac{F_{\pm2,\pm2,\pm1\pm1}}{\tilde{W}_{\pm2,\pm2}}(\varphi_{\pm1,\pm1}^{(1)})^2. \tag{20}$$

Substituting Eqs. (18)–(20) in Eq. (17), we obtain an equation for the quantities $\psi_l(y,t)=\epsilon\sqrt{q_l}\varphi_{l,l}^{(1)}$:

$$il\frac{\partial\psi_l}{\partial t}+\frac{\partial^2\psi_l}{\partial y_l^2}+|\psi_l|^2\psi_l=0,\quad l=n=\pm1, \tag{21}$$

where

$$q_l=\frac{M_l+L_l}{h_l},\quad y_l=\frac{z-v_gt}{\sqrt{p_l}},\quad h_l=ilh_{l,l},$$

$$p_l=-\frac{H_{l,l}}{h_lQ^2}=\frac{1}{2}\frac{\partial^2\Omega}{\partial Q^2},$$

$$M_l=(F_{l,l,-l,-l}+F_{l,l,2l,2l})\frac{F_{2l,2l,l,l}}{\tilde{W}_{2l,2l}},$$

$$L_l=\lambda_{l,l,-l,-l,l,l}+\lambda_{l,l,l,l,-l,-l}.$$

Equation (21) is the well-known NSE, which, under the condition $p_lq_l>0$, has the soliton solution

$$\psi_l=2il\eta\frac{e^{-il\varphi_{1_l}}}{\cosh 2\eta\varphi_{2_l}}, \tag{22}$$

where

$$\varphi_{1_l}=\frac{2\xi_0z}{\sqrt{p_l}}+2\left[2(\xi_0{}^2-\eta^2)-\frac{\xi_0v_g}{\sqrt{p_l}}\right]t-\varphi_0,$$

$$\varphi_{2_l}=\frac{z}{\sqrt{p_l}}+\left(4\xi_0-\frac{v_g}{\sqrt{p_l}}\right)t-y_0.$$

The quantities ξ_0, η, φ_0, and y_0 are scattering data, which arise when the NSE is solved by the inverse scattering transform (IST) [14,16]. Substituting the soliton solution (22) into Eq. (11), we obtain for the envelope $\hat{E}_l$ the breather solution [1–5,11,12]

$$\hat{E}_l=\frac{2il\eta}{\sqrt{q_l}}\frac{e^{-il(\varphi_{1_l}+\Omega t-Qz)}}{\cosh 2\eta\varphi_{2_l}}+O(\epsilon^2). \tag{23}$$

Using the IST, we can obtain the breather solution (23) for any initial value $\hat{E}(t=0,z)$. The appearance in Eq. (23) of the factor $e^{-il(Qz-\Omega t)}$ indicates the formation of periodic beats (slow in comparison with coordinates and time, with characteristic parameters Ω and Q), as a result of which the soliton solution (22) for $\varphi_{l,l}^{(1)}$ is transformed into the solution (23) for the envelope $\hat{E}_l$. Consequently, in dispersive and nonlinear media with quadratic and/or cubic nonlinearity, an optical nonlinear wave of the type of the breather (23) can propagate.

IV. BREATHER SOLUTION OF EQ. (10) IN THE SECOND CASE

Here we consider the same problem in Case II, i.e., we now investigate a boundary-value problem. In this case, we use expansion (12) for the solution of Eq. (10). On substituting Eq. (12) into Eq. (10), as was done in the preceding section, we obtain the NSE in the following form:

$$il\,\frac{\partial\chi_l}{\partial z}+\frac{\partial^2\chi_l}{\partial T_l^2}+|\chi_l|^2\chi_l=0,\quad l=\pm 1, \tag{24}$$

where

$$\chi_l=\epsilon\sqrt{\tilde{q}_l}f_{l,l}^{(1)},\quad T_l=\frac{1}{\sqrt{\tilde{p}_l}}\left(t-\frac{z}{u}\right),$$

$$\tilde{p}_l=\frac{\mu+\delta u^2-\gamma u}{u^2(\alpha+2\mu Q-\gamma\Omega)},\quad \tilde{q}_l=\frac{M_l+L_l}{\alpha+2\mu Q-\gamma\Omega}.$$

In this case, the relation between the quantities $\tilde{\Omega}$ and $\tilde{Q}$ has the same form as given by Eq. (18) if we substitute $\tilde{\Omega}$ and $\tilde{Q}$ for Ω and Q in the latter and

$$\chi_l=2il\eta\,\frac{e^{-il\delta_{1_l}}}{\cosh 2\eta\delta_{2_l}}, \tag{25}$$

where

$$\delta_{1_l}=\frac{2\xi_0}{\sqrt{\tilde{p}_l}}\,t+\left[4(\xi_0{}^2-\eta^2)-\frac{2\xi_0}{\sqrt{\tilde{p}_l}u}\right]z-\varphi_0,$$

$$\delta_{2_l}=\frac{t}{\sqrt{\tilde{p}_l}}+\left(4\xi_0-\frac{1}{u\sqrt{\tilde{p}_l}}\right)z-y_0.$$

By substituting Eq. (25) in Eq. (12), we obtain the breather solution of Eq. (10) in the second case:

$$\hat{E}_l=\frac{2il\eta}{\sqrt{\tilde{q}_l}}\,\frac{e^{-il(\delta_{1_l}+\tilde{\Omega}t-\tilde{Q}z)}}{\cosh 2\eta\delta_{2_l}}+O(\epsilon^2). \tag{26}$$

Using the IST, we can obtain the breather solution (26) of Eq. (10) for any boundary value of the quantity $\hat{E}(z=0,t)$.

V. THE STABILITY OF THE BREATHER

The wave equation (10) describes the situation where the propagation of nonlinear optical waves in a nonlinear dispersive medium is not influenced by such factors as, for example, the interaction of the optical radiation with impurity-resonance atoms contained in the medium or with thermal or coherent phonons, the effect of a finite conductivity, or others. Depending on the nature of their influence upon the wave process, these effects can be divided into two groups. The effects that lead to a change in the phase of the nonlinear wave enter into the first group, and the second group includes the effects causing the damping of breathers. We consider the effects associated with these two groups and influencing the breather by means of two examples: by the linear coherent interaction of optical radiation with impurity-resonance atoms contained in the medium, and by taking into account the conductivity. We can use perturbation theory, taking into account that the influence of these phenomena upon the breather is weak. For a systematic investigation of the influence of the effects mentioned above on the breather, it is necessary to begin with the introduction of the corresponding terms into the wave equation. In particular, the expression

$$4\pi\,\frac{d^2}{dt^2}\,(P_1+P_2) \tag{27}$$

must be added to Eq. (8), where [9,17,18]

$$P_1=\sum_{l=\pm 1}Z_l\hat{E}_l\left(i\,\frac{l\sigma n_0}{kc}\right), \tag{28}$$

with η_0 the index of refraction and σ the effective conductivity, takes into account the contribution to the polarization of the medium caused by the conductivity, while the quantity

$$P_2=n_0d_0s_1 \tag{29}$$

describes the effects of the one-photon resonance interaction of the optical pulse with the system of two-level impurity atoms that are contained in the medium, where n_0 is the concentration of optically active impurities and d_0 is the matrix element of the electric dipole moment of a two-level impurity atom. The dependence of the quantity P_2 on the strength of the electric field E is governed by the optical Bloch equations [10]

$$\frac{\partial s_1(t)}{\partial t}=-\omega_0s_2(t),$$

$$\frac{\partial s_2(t)}{\partial t}=\omega_0s_1(t)+\kappa_0E(t,z)s_3(t), \tag{30}$$

$$\frac{\partial s_3(t)}{\partial t}=-\kappa_0E(t,z)s_2(t),$$

where

$$\kappa_0=\frac{2d_0}{\hbar},\quad s_i(t)=\langle\hat{\sigma}_i(t)\rangle\ \ (\ i=1,2,3).$$

Here, $\langle\hat{\sigma}_i\rangle$ is the average value of the Pauli operator $\hat{\sigma}_i$, $\hbar$ is Planck's constant, and ω_0 is the frequency of the two-level atoms. In the interaction of an optical pulse with a resonantly absorbing medium, the most significant effects are usually observed at exact resonance. Therefore, for simplicity, we consider equations (30) at exact resonance, i.e., with $\omega=\omega_0$.

In the present section we will consider the solution of Eq. (8), taking into account the term (27) in Case I, under the condition

$$|Q_0|\ll 1, \tag{31}$$

where

$$Q_0(z,t)=\kappa_0\int_{-\infty}^{t}\hat{E}(z,t')dt'$$

is the area of the envelope of the optical pulse.

For the determination of the explicit form of the quantity P_2, we expand the quantity s_i in a perturbation-theory series in the small nonlinearity parameter ϵ,

$$s_i=\sum_{\alpha=0}^{\infty}\epsilon^{\alpha}B_i^{(\alpha)}.$$

Substituting this expansion and expression (11) in the set of equations (30), and taking into account Eqs. (28) and (29), we obtain

$$4\pi\frac{\partial^2 P_1}{\partial t^2}=-4\pi i\omega\sigma\epsilon^3\sum_{l,n} lZ_lY_n\varphi_{l,n}^{(1)}+O(\epsilon^4),$$
$$4\pi\frac{\partial^2 P_2}{\partial t^2}=\epsilon^3R\tau_0\sum_{l,n}\frac{l}{n}Z_lY_n\varphi_{l,n}^{(1)}+O(\epsilon^4), \tag{32}$$

where

$$R=\frac{4\pi n_0 d_0^2\omega^2}{\hbar\Omega},\quad \tau_0=\pm 1.$$

The plus sign corresponds to the initial condition, in which the impurity atoms are initially in the ground state, i.e., at $t\rightarrow-\infty$, $s_3=-1$ (attenuating medium). The minus sign corresponds to the case where at $t\rightarrow-\infty$, $s_3=+1$, i.e., all the impurity atoms are initially in the excited state (amplifying medium).

If we combine Eqs. (27) and (32) with Eq. (8), we can write the NSE in the following form:

$$il\left(\frac{\partial\psi_l}{\partial l}+\Gamma_l\psi_l\right)+r_l\psi_l+\frac{\partial^2\psi_l}{\partial y_l^2}+|\psi_l|^2\psi_l=0, \tag{33}$$

where

$$\Gamma_l=\frac{4\pi\omega\sigma}{h_l},\quad r_l=\frac{R}{h_l}\tau_0,\quad l=n=\pm 1.$$

We can remove the term $r_l\psi_l$ from this equation by changing the phase of the quantity ψ_l. Indeed, let us rewrite Eq. (33) for the quantity $\Theta_l=\psi_l e^{-ilr_lt}$:

$$il\left(\frac{\partial\Theta_l}{\partial t}+\Gamma_l\Theta_l\right)+\frac{\partial^2\Theta_l}{\partial y_l^2}+|\Theta_l|^2\Theta_l=0. \tag{34}$$

The soliton solution of this equation with $\Gamma_l=0$ is obtained by means of the IST in a manner analogous to the way in which Eq. (22) was obtained. Now it is more convenient to write the solution of Eq. (34) in the form

$$\Theta_l=\sqrt{2}K\frac{e^{il\Phi_{1_l}}}{\cosh\Phi_{2_l}}, \tag{35}$$

where

$$\Phi_{1_l}=\frac{v}{2\sqrt{p_l}}z-\left(\frac{v^2}{4}-K^2+\frac{vv_g}{2\sqrt{p_l}}\right)t,$$
$$\Phi_{2_l}=K\left[\frac{z}{\sqrt{p_l}}-\left(\frac{v_g}{\sqrt{p_l}}+v\right)t\right], \tag{36}$$

K is a parameter proportional to the amplitude of the soliton, and v is its velocity.

Using the results of the numerical calculations carried out in [15], we can assume that a good approximation to an exact solution of Eq. (34) will be expression (35) if the parameter K depends on time as

$$K_l(t)=K_l(0)e^{-2\Gamma_lt}. \tag{37}$$

By substituting Eq. (37) in Eqs. (35) and (36), and using Eq. (11), we can obtain a breather solution of Eq. (8), with Eq. (27) taken into account in the following form:

$$\hat{E}_l=\left(\frac{2}{q_l}\right)^{1/2}K_l(t)\frac{e^{i[r_lt+l(\Phi_{1_l}+Qz-\omega t)]}}{\cosh\Phi_{2_l}}+O(\epsilon^2). \tag{38}$$

In this expression, the quantities Φ_{1_l} and Φ_{2_l} contain the quantity $K_l(t)$ instead of K. Hence, Eq. (38) is a breather solution with damping $(\Gamma_l>0)$.

VI. CONCLUSION

In the present paper we have shown that in the propagation of intense optical radiation through a (quadratic and/or cubic) nonlinear and (spatially and/or temporally) dispersive medium, an optical breather can arise. The explicit form of the breather, when we consider the initial-value problem (Case I), is given by Eq. (23), and, if we investigate the boundary-value problem (Case II), the form of the breather is given by expression (26). The dispersion equation and connection between the quantities Ω and Q ($\tilde{\Omega}$ and $\tilde{Q}$) are given by the relations $W_{\pm 1}=0$ and Eq. (18).

The physical interpretation of the formation of a breather is the following. In the propagation of the pulse in a dispersive medium, its shape will not remain unchanged. The width of the pulse will increase during propagation. This is due to the fact that waves of different wavelength propagate in a dispersive wave with different velocities. In the NSE, this effect is taken into account through the terms $\partial^2\psi_l/\partial y_l^2$, $\partial^2\chi_l/\partial T^2$, $\partial^2\Theta_l/\partial y_l^2$.

On the other hand, the effects of nonlinearity lead to a progressive deformation of the profile of the pulse, which increases with increasing t. In the NSE, the nonlinear effects are taken into account by the terms $|\psi_l|^2\psi_l$, $|\chi_l|^2\chi_l$, $|\Theta_l|^2\Theta_l$.

As a result of the competition between the nonlinearity, which increases the curvature of the profile of the pulse, and the dispersion, which causes the profile to spread out, the shape of the nonlinear wave is stabilized; a breather state is formed.

It should be noted that our results and their interpretation are applicable to pulses with sufficiently smooth envelopes, under the condition that the size of the pulse is large in comparison with the wavelength, i.e., $kL\gg 1$. Moreover, the length of the breather should be significantly greater than the characteristic length of the periodic ''beats,'' $LQ \gg 1$, where L is the length of the breather.

We must note that in Secs. III and IV we have considered situations where $\varphi^{(1)}_{\pm 1,\pm 1} \neq 0$ and $f^{(1)}_{\pm 1,\pm 1} \neq 0$. Analogously, we can investigate situations where $\varphi^{(1)}_{\pm 1,\mp 1} \neq 0$ and $f^{(1)}_{\pm 1,\mp 1} \neq 0$.

In Sec. V, we investigated the stability of the breather relative to its interaction with resonance-impurity two-level atoms, and in the presence of a finite conductivity (in Case I, the initial value problem), and when $\Gamma_l > 0$.

Analogously, we can consider this question in Case II.

In Sec. V, we found that a linear-resonance interaction of the optical pulse with impurity atoms leads to a change in the phase of the pulse, and that the phase is positive or negative depending on whether we have the situation of attenuation (when $\tau_0 = 1$) or amplification (when $\tau_0 = -1$). It should be noted that this situation differs in principle from the situation of self-induced transparency, in which the interaction of the wave with the resonance atoms is essentially nonlinear.

The effects of conductivity reduce to the damping of the breather's amplitude according to an exponential law (37). If we consider other effects which lead to $\Gamma_l < 0$ (see, for example, [19,20]), then the amplitude of the breather will increase during propagation as $K_l(t) = K_l(0) \exp \Gamma_l t$. We note that the results of Sec. V are valid for breathers whose amplitude is small, as in Eq. (31). But at the same time, the results of Secs. III and IV are valid for breathers with any amplitude, because here we have not used the inequality (31).

The results of Sec. V are valid for situations where the effects of the linear interaction of the pulse with resonance atoms is of $O(\epsilon^3)$. If these effects are weaker, for example, of $O(\epsilon^4)$ or smaller, then we must use another method of solution of the NSE with damping terms (34). Such methods can be found, for example, in [21–23].

In Sec. V, we considered the case of exact resonance $\omega = \omega_0$ and homogeneous broadening of the spectral line. Extension to the case $\omega \neq \omega_0$ and inhomogeneous broadening of the spectral line do not present difficulties. It is obvious that in this case we should not expect qualitatively new results compared to those given above.

In conclusion, we note that the NSE contains not only Eq. (22) or (25) or (35), but also N-soliton solutions with a more complicated behavior. In particular, for many-soliton solutions of the NSE there are characteristic oscillations of the envelope and strong compression of the pulse peaks already in the initial stages of propagation of the wave. Under these conditions, we cannot always use the slowly varifying envelope approximation (9), and still less Eqs. (11) and (12) (the separation from $\hat{E}_l$ of the more slowly varying $\psi^{(\alpha)}_{l,n}$ and $f^{(\alpha)}_{l,n}$). Therefore, the scheme presented above is not valid for such solutions, and for that a completely different method is needed (see, for example, [6]).

ACKNOWLEDGMENT

The work of A.A.M. was supported in part by National Science Foundation through Grant No. DMR 93-19404.

[1] G. T. Adamashvili and S. V. Manakov, Solid State Commun. **48**, 381 (1983).
[2] G. T. Adamashvili, Opt. Spektrosk. **72**, 1400 (1992) [Opt. Spectrosc. **72**, 786 (1992)].
[3] G. T. Adamashvili, Opt. Spektrosk. **76**, 438 (1994) [Opt. Spectrosc. **76**, 391 (1994)].
[4] G. T. Adamashvili, L. V. Chkoniya, I. I. Chikaidze, and G. G. Nozadze, Kvant. Electron. (Moscow) **18**, 1251 (1991) [Sov. J. Quantum Electron. **21**, 1136 (1991)].
[5] A. C. Newell, *Solitoni v Mathematike i Physike* (Mir, Moscow, 1989) (in Russian).
[6] S. A. Ahmanov, V. A. Vislouch, and A. S. Chizkin, *Optika Phemtosekundnikh Lasernikh Impulsov* (Nauka, Moscow, 1988) (in Russian).
[7] M. V. Vinogradova, O. V. Rudenko, and A. P. Suhorukov, *Theoria Voln* (Nauka, Moscow, 1990) (in Russian).
[8] A. Yariv and P. Yeh, *Optical Waves in Crystals* (Wiley, New York, 1984).
[9] S. L. McCall and E. L. Hahn, Phys. Rev. **183**, 457 (1969).
[10] L. Allen and J. H. Eberly, *Opticheski Resonans i Dvukhurovnie Atomi* (Mir, Moscow, 1978) (in Russian).
[11] G. T. Adamashvili, Zh. Eksp. Teor. Fiz. **97**, 235 (1990) [Sov. Phys. JETP **70**, 131 (1990)].
[12] G. T. Adamashvili, Phys. Lett. A **164**, 90 (1992).
[13] *Nelineinaia Spektroskopia,* edited by N. Bloembergen (Mir, Moscow, 1979) (in Russian).
[14] R. K. Dodd, J. C. Eilbeck, J. D. Gibbon, and H. C. Morris, *Solitons and Nonlinear Wave Equation* (Academic, New York, 1982).
[15] T. Sukuma and Y. Kawanami, Phys. Rev. B **29**, 880 (1984).
[16] V. E. Zaharov, S. V. Manakov, S. P. Novikov, and L. P. Pitaevski, *Teoria Solitonov, Metod Obratnoi Zadachi* (Nauka, Moscow, 1980) (in Russian).
[17] D. J. Kaup, Phys. Rev. A **16**, 704 (1977).
[18] G. T. Adamashvili, Opt. Spektrosk. **80**, 251 (1996) [Opt. Spectrosc. **80**, 219 (1996)].
[19] A. Hasegawa and Y. Kadama, Opt. Lett. **7**, 285 (1982).
[20] Y. Kadama and A. Hasegawa, Opt. Lett. **7**, 339 (1982).
[21] D. J. Kaup and A. C. Newell, Proc. R. Soc. London, Ser. A **361**, 413 (1978).
[22] A. C. Newell, in *Solitoni*, edited by S. P. Novikov (Mir, Moscow, 1983) (in Russian).
[23] D. J. Kaup, SIAM J. Appl. Math. **31**, 121 (1976).